# 16 The Side-Angle-Side Axiom

**1.** (a) Let $\overline{MN}$ and $\overline{PQ}$ denote two given line segments. Explain what it mean that $\overline{MN} \cong \overline{PQ}$.

(b) Let $\measuredangle ABC$ and $\measuredangle DEF$ denote given angles. Explain, what it mean that $\measuredangle ABC \cong \measuredangle DEF$.

Intuitively, two figures are congruent if one can be "picked up and laid down exactly on the other" so that the two coincide.

**Convention**. In $\triangle ABC$, if there is no confusion, we will denote $\measuredangle CAB$ by $\measuredangle C$, $\measuredangle ABC$ by $\measuredangle B$ and $\measuredangle BCA$ by $\measuredangle C$.

**Definition (congruence, congruent triangles)**

Let $\triangle ABC$ and $\triangle DEF$ be two triangles in a protractor geometry and let $f : \{A,B,C\} \to \{D,E,F\}$ be a bijection between the vertices of the triangles. $f$ is a congruence iff

$$\overline{AB} = \overline{f(A)f(B)}, \qquad \overline{BC} = \overline{f(B)f(C)}, \qquad \overline{CA} = \overline{f(C)f(A)},$$

$$\measuredangle A \cong \measuredangle f(A), \qquad \measuredangle B \cong \measuredangle f(B) \qquad \text{and} \qquad \measuredangle C \cong \measuredangle f(C).$$

Two triangles, $\triangle ABC$ and $\triangle DEF$, are congruent if there is a congruence $f : \{A,B,C\} \to \{D,E,F\}$. If the congruence is given by $f(A) = D$, $f(B) = E$, and $f(C) = F$, then we write $\triangle ABC \cong \triangle DEF$.

**2.** Prove that congruence is an equivalence relation on the set of all triangles in a protractor geometry.

The fundamental question of this section is: How much do we need to know about a triangle so that it is determined up to congruence?

Suppose that we are given $\triangle ABC$ and a ray $\overrightarrow{EX}$ which lies on the edge of a half plane $H_1$. Then we can construct the following by the Segment Construction Theorem and the Angle Construction Theorem

(a) A unique point $D \in \overrightarrow{EX}$ with $\overline{BA} \cong \overline{ED}$;

(b) A unique ray $\overrightarrow{EY}$ with $Y \in H_1$ and $\measuredangle ABC \cong \measuredangle XEY$;

(c) A unique point $F \in \overrightarrow{EY}$ with $\overline{BC} \cong \overline{EF}$.

Is $\triangle ABC \cong \triangle DEF$? Intuitively it should be (and it will be if SAS is satisfied). However, since we know nothing about the rulers for $\overleftrightarrow{DF}$ and $\overleftrightarrow{AC}$, we have no way of showing that $\overline{AC} \cong \overline{DF}$. In fact next example will show that $\overline{AC}$ need not be congruent to $\overline{DF}$.

**3.** In the Taxicab Plane let $A(1,1)$, $B(0,0)$, $C(-1,1)$, $E(0,0)$, $X(3,0)$, and let $H_1$ be the half plane above the $x$-axis. Carry out the construction outlined above and check to see whether or not $\triangle ABC$ is congruent to $\triangle DEF$.

[Example 6.1.1, page 126]

**Definition (Side-Angle-Side Axiom (SAS))**

A protractor geometry satisfies the Side-Angle-Side Axiom (SAS) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\overline{AB} \cong \overline{DE}$, $\measuredangle B \cong \measuredangle E$ and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

**Definition (neutral or absolute geometry)**

A neutral geometry (or absolute geometry) is a protractor geometry which satisfies SAS.

**Proposition (Euclidean Law of Cosines)**. Let $c(\theta)$ be the cosine function as developed in Section 15. Then for any $\triangle PQR$ in the Euclidean Plane $d_E(P,R)^2 = d_E(P,Q)^2 + d_E(Q,R)^2 - 2d_E(P,Q)d_E(Q,R)\,c(m_E(\measuredangle PQR))$.

**Proposition**. The Euclidean Plane $\mathcal{E}$ satisfies SAS.

**4.** Prove the above Proposition.

[Proposition 6.1.3, page 128]

**Proposition**. The Poincaré Plane $\mathbb{H}$ is a neutral geometry.

**Definition (isosceles triangle, scalene triangle, equilateral triangle, base angles)** A triangle in a protractor geometry is isosceles if (at least) two sides are congruent. Otherwise, the triangle is scalene. The triangle is equilateral if all three sides are congruent. If $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{BC}$, then the base angles of $\triangle ABC$ are $\measuredangle A$ and $\measuredangle C$.

Our first application of SAS is the following theorem on isosceles triangles. The Latin name (literally "the bridge of asses") refers to the complicated figure Euclid used in his proof, which looked like a bridge, and to the fact that only someone as dull as an ass would fail to understand it.

**Theorem**. **(Pons Asinorum)**. In a neutral geometry, the base angles of an isosceles triangle are congruent.

**5.** Prove the above Theorem.

[Theorem 6.1.5, page 129]

**6.** Let $\triangle ABC$ be an isosceles triangle in a neutral geometry with $\overline{AB} \cong \overline{CA}$. Let $M$ be the midpoint of $\overline{BC}$. Prove that $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$.

**7.** Prove that in a neutral geometry every equilateral triangle is **equiangular**; that is, all its angles are congruent.

**8.** Show that if $\triangle ABC$ is a triangle in the Euclidean Plane which has a right angle at $C$ then $(AB)^2 = (AC)^2 + (BC)^2$.

**9.** Let $\triangle ABC$ be a triangle in the Euclidean Plane with $\measuredangle C$ a right angle. If $m_E(\measuredangle B) = \theta$ prove that $c(\theta) = BC/AB$ and $s(\theta) = AC/AB$.

**10.** Let $\square ABCD$ be a quadrilateral in a neutral geometry with $\overline{CD} \cong \overline{CB}$. If $\overrightarrow{CA}$ is the bisector of $\measuredangle DCB$ prove that $\overline{AB} \cong \overline{AD}$.

**11.** Let $\square ABCD$ be a quadrilateral in a neutral geometry and assume that there is a point $M \in \overline{BD} \cap \overline{AC}$. If $M$ is the midpoint of both $\overline{BD}$ and $\overline{AC}$ prove that $\overline{AB} \cong \overline{CD}$.

**12.** Suppose there are points $A$, $B$, $C$, $D$, $E$ in a neutral geometry with $A-D-B$ and $A-E-C$ and $A$, $B$, $C$ not collinear. If $\overline{AD} \cong \overline{AE}$ and $\overline{DB} \cong \overline{EC}$ prove that $\measuredangle EBC \cong \measuredangle DCB$.

## 17 Basic Triangle Congruence Theorems

**Definition**. **(Angle-Side-Angle Axiom (ASA))** A protractor geometry satisfies the Angle-Side-Angle Axiom (ASA) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\measuredangle A \cong \measuredangle D$, $\overline{AB} \cong \overline{DE}$, and $\measuredangle B \cong \measuredangle E$, then $\triangle ABC \cong \triangle DEF$.

**Theorem**. A neutral geometry satisfies ASA.

**1.** Prove the above Theorem.

[Theorem 6.2.1, page 131]

**Theorem**. **(Converse of Pons Asinorum)**. In a neutral geometry, given $\triangle ABC$ with $\measuredangle A \cong \measuredangle C$, then $\overline{AB} \cong \overline{CB}$ and the triangle is isosceles.

**2.** Prove the above Theorem.

**3.** Prove that in a neutral geometry every equiangular triangle is also equilateral.

**Definition**. **(Side-Side-Side Axiom (SSS))** A protractor geometry satisfies the Side-Side-Side Axiom (SSS) if whenever $\triangle ABC$ and $\triangle DEF$ are two triangles with $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

**Theorem**. A neutral geometry satisfies SSS.

**4.** Prove the above Theorem.

[Theorem 6.2.3, page 132]

In one of earlier sections we showed that PSA and PP are equivalent axioms: if a metric geometry satisfies one of them then it also satisfies the other. A similar situation is true for SAS and ASA. We already know that SAS implies ASA. The next theorem gives the converse.

**Theorem**. If a protractor geometry satisfies ASA then it also satisfies SAS and is thus a neutral geometry.

**5.** Prove the above Theorem.

**Theorem**. In a neutral geometry, given a line $\ell$ and a point $B \notin \ell$, then there exists at least one line through $B$ perpendicular to $\ell$.

**6.** Prove the above Theorem.

[Theorem 6.2.5, page 133]

**7.** In a neutral geometry, given $\triangle ABC$ with $\overline{AB} \cong \overline{BC}$, $A-D-E-C$, and $\measuredangle ABD \cong \measuredangle CBE$, prove that $\overline{DB} \cong \overline{EB}$.

**8.** In a neutral geometry, given $\triangle ABC$ with $A-D-E-C$, $\overline{AD} \cong \overline{EC}$, and $\measuredangle CAB \cong \measuredangle ACB$, prove that $\measuredangle ABE \cong \measuredangle CBD$.

**9.** In a neutral geometry, given $\square ABCD$ with $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, prove that $\measuredangle A \cong \measuredangle C$ and $\measuredangle B \cong \measuredangle D$.

**10.** In a neutral geometry, given $\triangle ABC$ with $A-D-B$, $A-E-C$, $\measuredangle ABE \cong \measuredangle ACD$, $\measuredangle BDC \cong \measuredangle BEC$, and $\overline{BE} \cong \overline{CD}$, prove that $\triangle ABC$ is isosceles.

# Aksiom stranica-ugao-stranica

# (a) Date su dvije duži $\overline{MN}$ i $\overline{PQ}$. Obrazložiti šta znači da je $\overline{MN} \cong \overline{PQ}$

(b) Data su dva ugla $\measuredangle ABC$ i $\measuredangle DEF$. Obrazložiti šta znači $\measuredangle ABC \cong \measuredangle DEF$.

Rj.

(a) $\overline{MN} \stackrel{def.}{=} \{ A \in \mathcal{S} \mid M-A-N \text{ ili } A=M \text{ ili } A=N \}$

$\overline{PQ} \stackrel{def}{=} \{ B \in \mathcal{S} \mid P-B-Q \text{ ili } B=P \text{ ili } Q=B \}$

$$\overline{MN} \cong \overline{PQ} \iff MN = PQ$$

$MN = d(M,N)$ — dužina duži $\overline{MN}$

$PQ = d(P,Q)$ — dužina duži $\overline{PQ}$

(b)

$\measuredangle ABC \stackrel{def}{=} \overrightarrow{BA} \cup \overrightarrow{BC}$

$= pp[B,A) \cup pp[B,C)$

$\overrightarrow{BC} \stackrel{def}{=} \overline{BC} \cup \{ R \in \mathcal{S} \mid B-C-R \}$

$\measuredangle DEF \stackrel{def}{=} \overrightarrow{ED} \cup \overrightarrow{EF}$

$$\measuredangle ABC \cong \measuredangle DEF \iff m(\measuredangle ABC) = m(\measuredangle DEF)$$

šta znači $m(\measuredangle ABC)$ smo objasnili u Protractor postulatu (tj. u definiciji mjere ugla)

## Dogovor

U trouglu $\Delta ABC$, ako ne postoji mogućnost za zabunu, ugao $\measuredangle ABC$ ćemo označiti sa $\measuredangle B$, ugao $\measuredangle CAB$ sa $\measuredangle A$ i $\measuredangle BCA$ sa $\measuredangle C$. Drugim riječima

$$\measuredangle A = \measuredangle CAB, \qquad \measuredangle B = \measuredangle ABC, \qquad \measuredangle C = \measuredangle BCA.$$

## Definicija (podudarnost)

Neka su $\Delta ABC$ i $\Delta DEF$ dva trougla u protractor geometriji i neka je $f: \{A, B, C\} \to \{D, E, F\}$ bijekcija između vrhova trouglova. Bijekcija $f$ je podudarnost ako

$$\overline{AB} \cong \overline{f(A)f(B)}, \qquad \overline{BC} \cong \overline{f(B)f(C)}, \qquad \overline{CA} \cong \overline{f(C)f(A)}$$

i

$$\measuredangle A \cong \measuredangle f(A), \qquad \measuredangle B \cong \measuredangle f(B), \qquad \measuredangle C \cong \measuredangle f(C).$$

Dva trougla $\Delta ABC$ i $\Delta DEF$ su podudarna ako postoji podudarnost $f: \{A, B, C\} \to \{D, E, F\}$. Ako je podudarnost data sa $f(A) = D$, $f(B) = E$ i $f(C) = F$, tada pišemo $\Delta ABC \cong \Delta DEF$.

(#) Pokazati da je podudarnost relacija ekvivalencije na skupu svih trouglova u protractor geometriji.

Rj.

REFLEKSIVNOST

Posmatrajmo bijekciju $f:\{A,B,C\}\to\{A,B,C\}$ definisana sa $f(A)=A$, $f(B)=B$, $f(C)=C$. Tada

$$\overline{AB}\cong\overline{f(A)f(B)},\quad \overline{BC}\cong\overline{f(B)f(C)},\quad \overline{CA}\cong\overline{f(C)f(A)},$$

$$\measuredangle A\cong\measuredangle f(A),\quad \measuredangle B\cong\measuredangle f(B)\quad ;\quad \measuredangle C\cong f(C) \qquad \ldots (1)$$

Drugim riječima $f$ je podudarnost i imamo $\Delta ABC\cong\Delta ABC$.

SIMETRIČNOST

Pretpostavimo da je $\Delta ABC\cong\Delta PQR$. To znači da postoji bijekcija $f:\{A,B,C\}\to\{P,Q,R\}$ za koju vrijedi svih šest uslova iz (1) i za koju je $f(A)=P$, $f(B)=Q$, $f(C)=R$. Sad možemo definisati bijekciju $g:\{P,Q,R\}\to\{A,B,C\}$ na sljedeći način $g(P)=A$, $g(Q)=B$, $g(R)=C$. Tada

$$\overline{PQ}\cong\overline{g(P)g(Q)},\quad \overline{QR}\cong\overline{g(Q)g(R)},\quad \overline{RP}\cong\overline{g(R)g(P)}$$

npr. $\overline{g(Q)g(R)}=\overline{BC}$
a na osnovu (1) imamo da $\overline{BC}\cong\overline{QR}$

$$\measuredangle P\cong\measuredangle g(P),\quad \measuredangle Q\cong\measuredangle g(Q)\quad i\quad \measuredangle R\cong\measuredangle g(R)$$

Drugim riječima $\Delta PQR\cong\Delta ABC$.

TRANZITIVNOST

Za vježbu!

Pretpostavimo da je dat $\Delta ABC$ i da je data poluprava pp$[E, X) = \overrightarrow{EX}$ koja leži u ivici poluravni $H_1$.
Prisjetimo se Konstrukcija duži teoreme i Konstrukcija ugla teoreme:

Konstrukcija duži teorema: Ako je pp$[A,B) = \overrightarrow{AB}$ data poluprava i ako je $\overline{CD}$ data duž u metričkoj geometriji, tada postoji jedinstvena tačka $C \in \overrightarrow{AB}$ taka da $\overline{PQ} \cong \overline{AC}$.

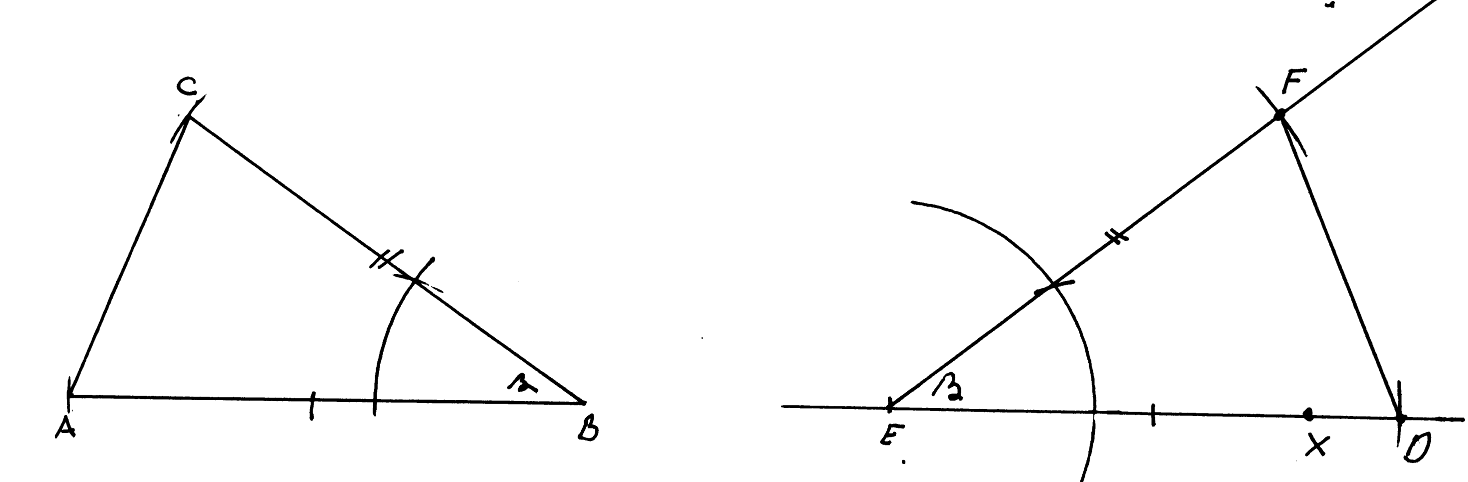
Konstrukcija ugla teorem: U protractor geometriji, za dati ugao $\measuredangle ABC$ i polupravu $\overrightarrow{ED}$ koja pripada ivici poluravni $H_1$, postoji jedinstvena poluprava pp$[E,F) = \overrightarrow{EF}$ taka da $F \in H_1$ i $\measuredangle ABC \cong \measuredangle DEF$.

Koristeći ove dvije teoreme konstruišimo trougao $\Delta DEF$ na sljedeći način

(a) Jedinstvenu tačku $D \in \overrightarrow{EX}$ sa osobinom da $\overline{BA} \cong \overline{ED}$.

(b) Jedinstvenu polupravu pp$[E,Y) = \overrightarrow{EY}$ gdje $Y \in H_1$ i $\measuredangle ABC \cong \measuredangle XEY$

(c) Jedinstvenu tačku $F \in \overrightarrow{EY}$ sa osobinom $\overline{BC} \cong \overline{EF}$.




Da li je $\Delta ABC \cong \Delta DEF$? Intuitivno, trebalo bi biti (i jest ako je aksiom SUS zadovoljen). Međutim, kako ne znamo ništa o mjerama za $\overleftrightarrow{DF}$ i $\overleftrightarrow{AC}$, nemamo načina da pokažemo da je $\overline{AC} \cong \overline{DF}$. U stvari, sljedeći primjer će pokazati da $\overline{AC}$ nemora biti podudarno sa $\overline{DF}$,

#) U Taksi ravni date su tačke $A(1,1)$, $B(0,0)$, $C(-1,1)$, $E(0,0)$, $X(3,0)$ i neka je $H_1$ poluravan iznad x-ose. Sprovedimo konstrukciju opisanu prije zadatka i provjerimo da li je ili nije $\Delta ABC$ podudaran sa $\Delta DEF$.

Rj. Prisjetimo se $d_T(B,A) = |f(B) - f(A)|$

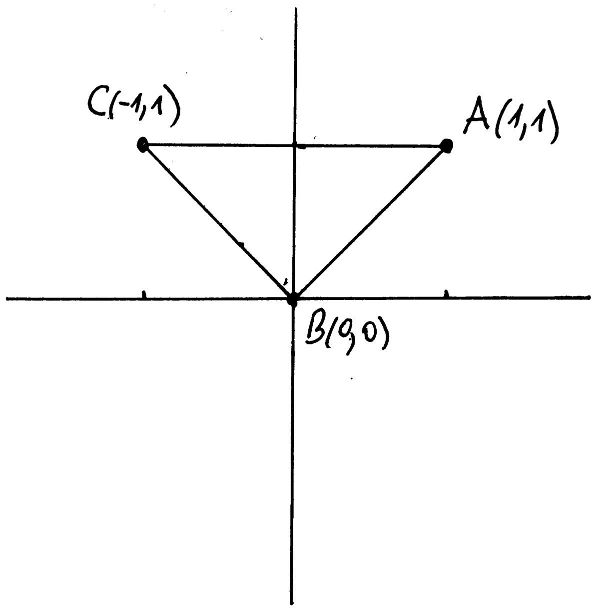
Koordinatni sistem za $L_a$ je $f(a,y)=y$

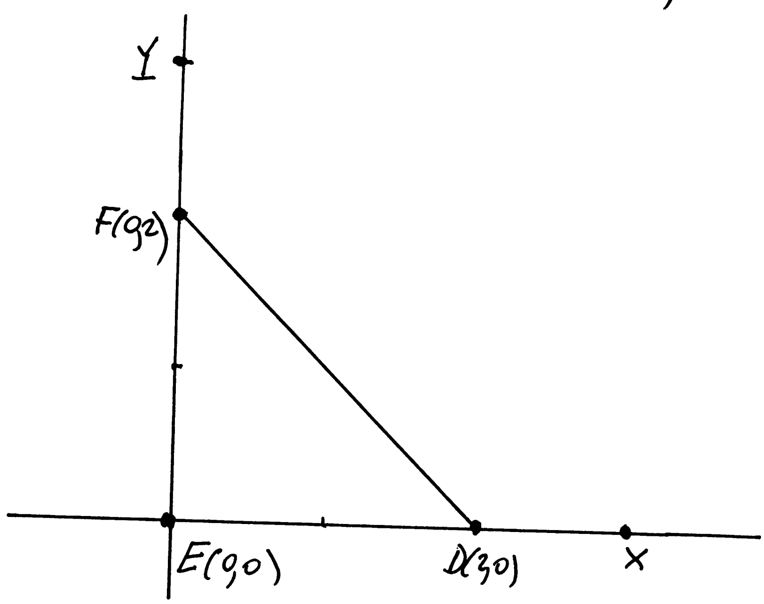
—||— za $L_{m,b}$ je $f(x,y) = (1+|m|)x$

Prisjetimo se da je također

$$d_T(P,Q) = |x_1-x_2|+|y_1-y_2|, \quad P(x_1,y_1),\ Q(x_2,y_2)$$

$BA = 1+1=2 \Rightarrow D(2,0)$




$$m_E(\measuredangle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\cdot\|C-B\|}\right) = \cos^{-1}\left(\frac{\langle(1,1),(-1,1)\rangle}{\|(1,1)\|\cdot\|(-1,1)\|}\right) = \cos^{-1}\frac{-1+1}{\sqrt{2}\cdot\sqrt{2}} = 90$$

$m_E(\measuredangle ABC) = 90 \Rightarrow Y=(0,3)$

$BC = 1+1=2 \Rightarrow F(0,2)$

Taxi i Euklidova mjera ugla su iste pa je:

$$m_E(\measuredangle BCA) = \cos^{-1}\frac{\langle B-C, C-A\rangle}{\|B-C\|\cdot\|C-A\|} = \cos^{-1}\frac{\langle(1,-1),(-2,0)\rangle}{\|(1,-1)\|\cdot\|(-2,0)\|} = \cos^{-1}\frac{-2}{\sqrt{2}\cdot 2} = 45$$

$m_E(\measuredangle BCA) = 45 = m_E(\measuredangle EFD) \quad \Rightarrow \quad \measuredangle BCA \cong \measuredangle EFD$

$m_E(\measuredangle CAB) = 45 = m_E(\measuredangle FDE) \quad \Rightarrow \quad \measuredangle CAB \cong \measuredangle FDE$

S druge strane

$d_T(A, C) = 2 \quad ; \quad d_T(D, F) = 2 + 2 = 4 \quad \Rightarrow \quad \overline{AC} \not\cong \overline{DF}$

Prema tome $\Delta ABC$ nije podudaran sa $\Delta DEF$.

Definicija (SUS aksiom)

Protractor geometrija zadovoljava stranica-ugao-stranica aksiom (SUS) ako za bilo koja dva trougla $\Delta ABC$ i $\Delta DEF$ sa osobinama $\overline{AB} \cong \overline{DE}$, $\measuredangle B \cong \measuredangle E$ i $\overline{BC} \cong \overline{EF}$ imamo da tada vrijedi $\Delta ABC \cong \Delta DEF$.

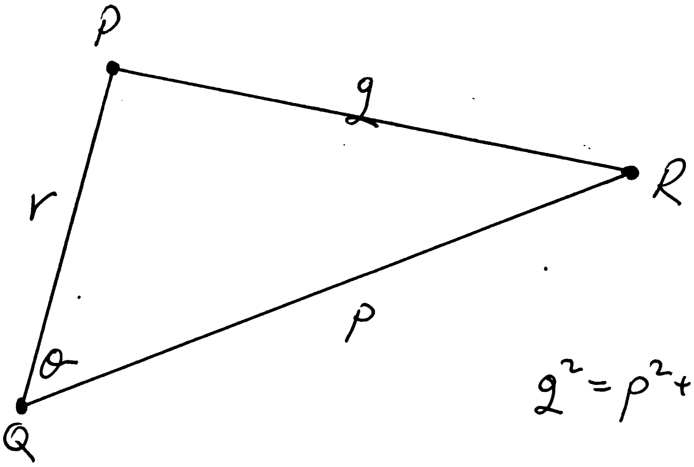
Definicija (neutralna ili apsolutna geometrija)

Neutralna geometrija (ili apsolutna geometrija) je protractor geometrija koja zadovoljava SUS aksiom.

Propozicija (Euklidov zakon kosinusa)

Neka je $c(\theta)$ cosinus f-ja. Tada za proizvoljan $\Delta PQR$ u Euklidovoj ravni

$$d_E(P,R)^2 = d_E(P,Q)^2 + d_E(Q,R)^2 - 2\, d_E(P,Q)\, d_E(Q,R)\, c(m_E(\measuredangle PQR))$$




$$q^2 = p^2 + r^2 - 2pr\cos(\theta)$$

Propozicija

Euklidova ravan $\mathcal{E}$ zadovoljava SUS.

(#) Dokazati propoziciju iznad.

Rj. Dokaz se nalazi u knjizi. (Propozicija 6.1.3, str. 128)

Skica dokaza

$\triangle ABC$, $\triangle DEF$ dati tr. t.d. $\overline{AB} \cong \overline{DE}$, $\sphericalangle B \cong \sphericalangle E$, $\overline{BC} \cong \overline{EF}$.

Euklid. zak. kosinusa $\Rightarrow$ $(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(AC)\, c(m_E(\sphericalangle B))$

$$= (DE)^2 + (EF)^2 - 2(DE)(EF)\, c(m_E(\sphericalangle E))$$

$$= (DF)^2$$

Time je $AC = DF$ pa je $\overline{AC} \cong \overline{DF}$.

Prema Eukl. zak kosin. za proizv. $\triangle PQR$ imamo

$$PR^2 = PQ^2 + QR^2 - 2 \cdot PQ \cdot QR \cdot c(m_E(\sphericalangle PQR)) \Rightarrow c(m_E(\sphericalangle PQR)) = \frac{PQ^2 + QR^2 - PR^2}{2PQ \cdot QR}$$

U specijalnom slučaju

$$c(m_E(\sphericalangle BAC)) = \frac{BA^2 + AC^2 - BC^2}{2 \cdot BA \cdot AC} = \frac{ED^2 + DF^2 - EF^2}{2 \cdot ED \cdot DF} = c(m_E(\sphericalangle EDF))$$

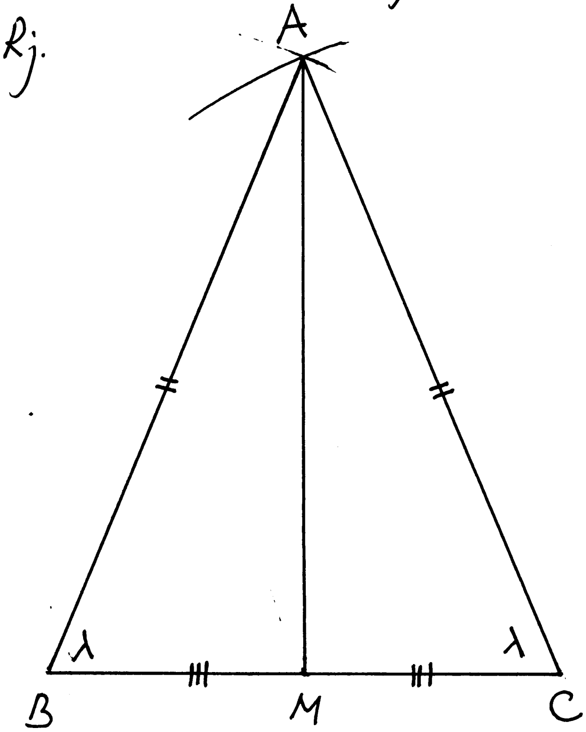
Kako je $c(\theta)$ injektivna (za $0 < \theta < 180$)

$$m_E(\sphericalangle BAC) = m_E(\sphericalangle EDF) \quad i \quad \sphericalangle A \cong \sphericalangle D.$$

Slično bi pokazali da je $\sphericalangle C \cong \sphericalangle F$ pa je $\sphericalangle ABC \cong \triangle DEF$.

(#) Neka je $\Delta ABC$ jkk trougao u neutralnoj geometriji takav da $\overline{AB} \cong \overline{CA}$ i neka je $M$ sredina stranice $\overline{BC}$. Dokazati da je $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$.

Rj.




$$\Delta ABC \text{ jkk} \xRightarrow{\text{Teor. Pons. Asinorum}} \measuredangle ABC \cong \measuredangle ACM$$

Neka je $M$ sredina stranice $BC$.
Označimo uglove $\measuredangle ABM$ i $\measuredangle ACM$ sa $\lambda$.

$$\left.\begin{array}{l} AB \cong AC \\ \measuredangle ABM \cong \measuredangle ACM \\ BM \cong CM \end{array}\right\} \xrightarrow{SUS} \Delta ABM \cong \Delta ACM$$

$$\Downarrow$$

$$\measuredangle AMB \cong \measuredangle AMC \quad \dots(1)$$

Prisjetimo se Teoreme linearnog para:
Ako uglovi $\measuredangle ABC$ i $\measuredangle CBD$ formiraju linearan par u protractor geometriji tada su oni suplementarni

Drugim riječima $\measuredangle AMB + \measuredangle AMC = 180 \xRightarrow{(1)} \measuredangle AMB = 90^\circ = \measuredangle AMC$

$$\Downarrow$$

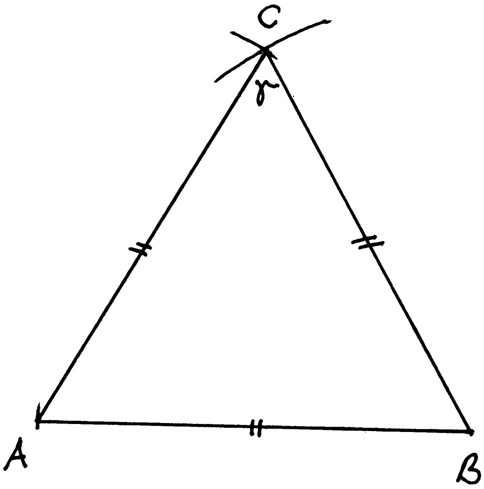
$$\overline{AM} \perp \overline{BC}$$

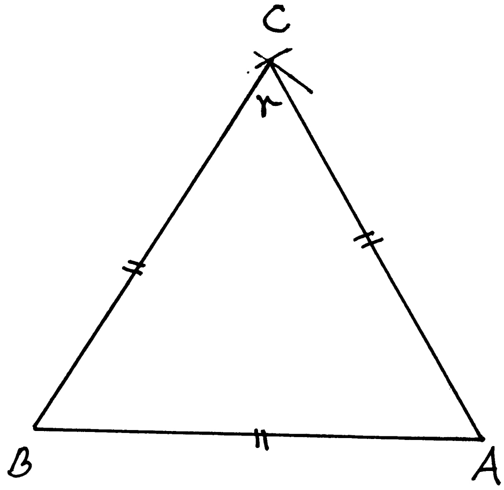
$$\Downarrow$$

$$\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$$

(#) Pokazati da su svi uglovi u jednakostraničnom trouglu podudarni.

Rj. Prvo želim pokazati da je $\sphericalangle ABC \cong \sphericalangle CAB$. Za tu svrhu nacrtajmo $\triangle ABC$ na dva različita načina.



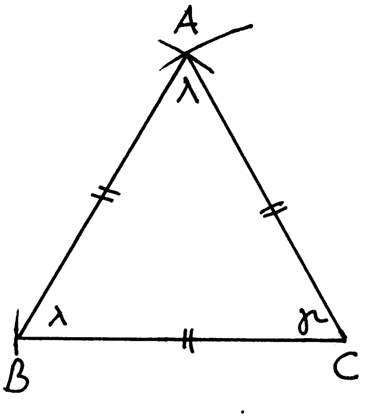

Označimo $\sphericalangle ACB$ sa $\gamma$. i pokažimo da je $\triangle ACB \cong \triangle BCA$.

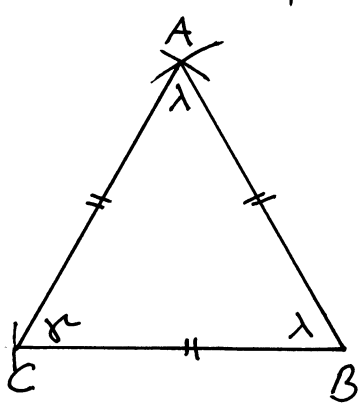
$$\left.\begin{array}{c} \overline{AC} \cong \overline{BC} \\ \sphericalangle ACB \cong \sphericalangle BCA = \gamma \\ \overline{CB} \cong \overline{CA} \end{array}\right\} \xRightarrow{SUS} \triangle ACB \cong \triangle BCA$$

$$\Downarrow$$

$$\sphericalangle CAB \cong \sphericalangle CBA$$

Označimo ova dva ugla sa $\lambda$ i pokažimo da je $\lambda = \gamma$.
Posmatrajmo $\triangle ABC$ nacrtan na sljedeća dva načina




$$\left.\begin{array}{c} \overline{AB} \cong \overline{AC} \\ \sphericalangle BAC \cong \sphericalangle CAB = \lambda \\ \overline{AC} \cong AB \end{array}\right\} \xRightarrow{SUS} \triangle BAC \cong \triangle CAB$$

$$\Downarrow$$

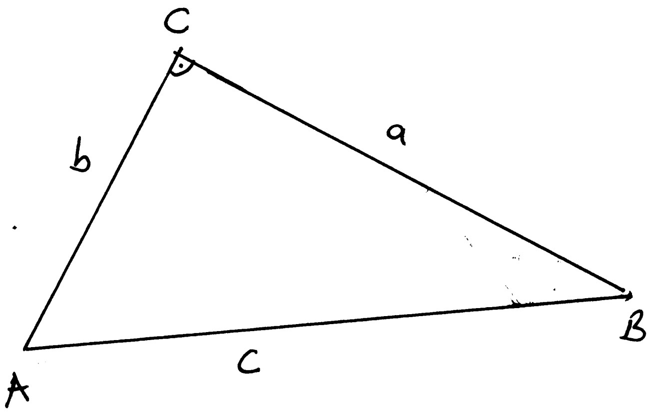
$$\sphericalangle ABC \cong \sphericalangle ACB \Rightarrow \lambda = \gamma$$

q.e.d.

# Ako je $\Delta ABC$ trougao u Euklidovoj ravni koji ima prav ugao kod vrha C pokazati da je tada

$$(AB)^2=(AC)^2+(BC)^2.$$

Rj.




Uvedimo oznake $\overline{AB}=c$, $\overline{BC}=a$, $\overline{AC}=b$.

Prema Euklidovom zakonu kosinusa

$$c^2=a^2+b^2-2ab\,c(m_E(\measuredangle ACB))$$

Kako je $m_E(\measuredangle ACB)=90$ to je $m_E(\measuredangle C)=0$.

Drugim riječima imamo da

$$c^2=a^2+b^2$$

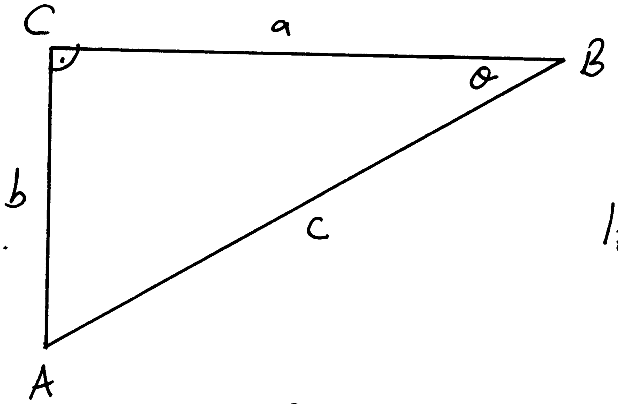
$$\Downarrow$$

$$(AB)^2=(BC)^2+(AC^2)$$

q.e.d.

(#) Neka je $\Delta ABC$ trougao u Euklidovoj ravni sa pravim uglom kod vrha $\measuredangle C$. Ako je $m_E(\measuredangle B) = \theta$ dokazati da $c(\theta) = \dfrac{BC}{AB}$ i $s(\theta) = \dfrac{AC}{AB}$.

Rj.




Uvedimo oznake $\overline{AC} = b$, $\overline{AB} = c$ i $\overline{BC} = a$.

Iz prethodnog zadatka

$$c^2 = a^2 + b^2$$

Prema Euklidovom zakonu cosinusa (cosinusna teorema)

$$b^2 = a^2 + c^2 - 2ac \cdot c(m_E(\measuredangle B))$$

$$2ac \cdot c(\theta) = a^2 + \underbrace{c^2}_{a^2+b^2} - b^2$$

$$c(\theta) = \frac{2a^2}{2ac} \Rightarrow c(\theta) = \frac{a}{c} \Rightarrow c(\theta) = \frac{BC}{AB}$$

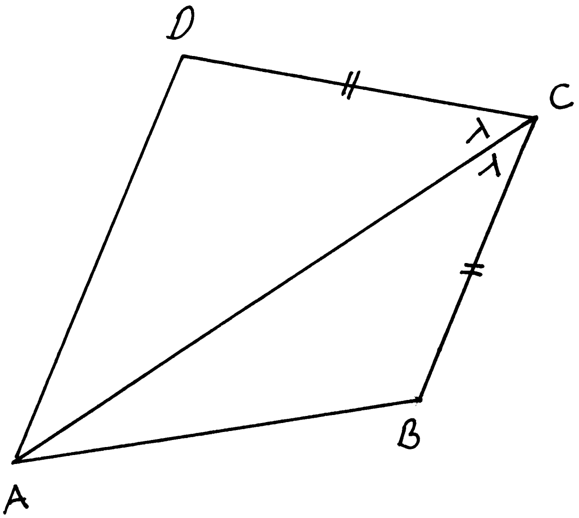
S obzirom da je $s^2(\theta) + c^2(\theta) = 1$

$$s^2(\theta) = 1 - \frac{a^2}{c^2} = \frac{c^2 - a^2}{c^2} = \frac{b^2}{c^2}$$

$$s(\theta) = \frac{b}{c} \Rightarrow s(\theta) = \frac{AC}{AB}.$$

\# Neka je $\square ABCD$ četverougao u neutralnoj geometriji za koji vrijedi da je $\overline{CD} \cong \overline{CB}$. Ako je $\overrightarrow{CA}$ simetrala ugla $\sphericalangle DCB$ pokazati da je tada $\overline{AB} \cong \overline{AD}$.

Rj.:




$\overrightarrow{CA}$ sim. $\sphericalangle DCB \Rightarrow \sphericalangle DCA \cong \sphericalangle BCA$

Uvedimo oznaku $\lambda$ za ova dva ugla.

$(m(\sphericalangle ACD) = \lambda \Rightarrow m(\sphericalangle ACB) = \lambda)$

Posmatrajmo $\triangle ACD$ i $\triangle ACB$

$$\left.\begin{array}{l} \overline{AC} \cong \overline{AC} \\ \sphericalangle ACD \cong \sphericalangle ACB \\ \overline{CD} \cong \overline{CB} \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle ACD \cong \triangle ACB$$

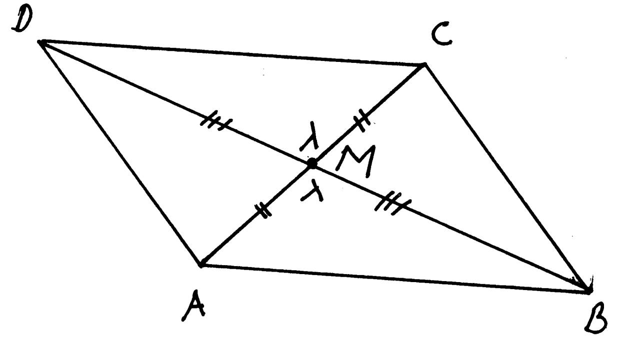
$$\Downarrow$$

$$\overline{AD} \cong \overline{AB}$$

q.e.d.

(#) Neka je $\square ABCD$ četvorougao u neutralnoj geometriji i pretpostavimo da postoji tačka $M \in \overline{BD} \cap \overline{AC}$. Ako je $M$ sredina obe dijagonale $\overline{BD}$ i $\overline{AC}$ pokazati da je $\overline{AB} \cong \overline{CD}$.

Rj.




Primetimo da su $\measuredangle DMC$ i $\measuredangle AMB$ unakrsni uglovi. U jednoj od prethodnih teorema smo pokazali da su unakrsni uglovi podudarni.

$$\Downarrow$$

$$\measuredangle DMC \cong \measuredangle BMA$$

(Pa ako je $m(\measuredangle DMC) = \lambda \Rightarrow m(\measuredangle AMB) = \lambda$)

Sad imamo

$$\left.\begin{array}{c} \overline{MD} \cong \overline{MB} \\ \measuredangle DMC \cong \measuredangle BMA \\ \overline{MC} \cong \overline{MA} \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle DMC \cong \triangle BMA$$

$$\Downarrow$$

$$\overline{DC} \cong \overline{BA}$$

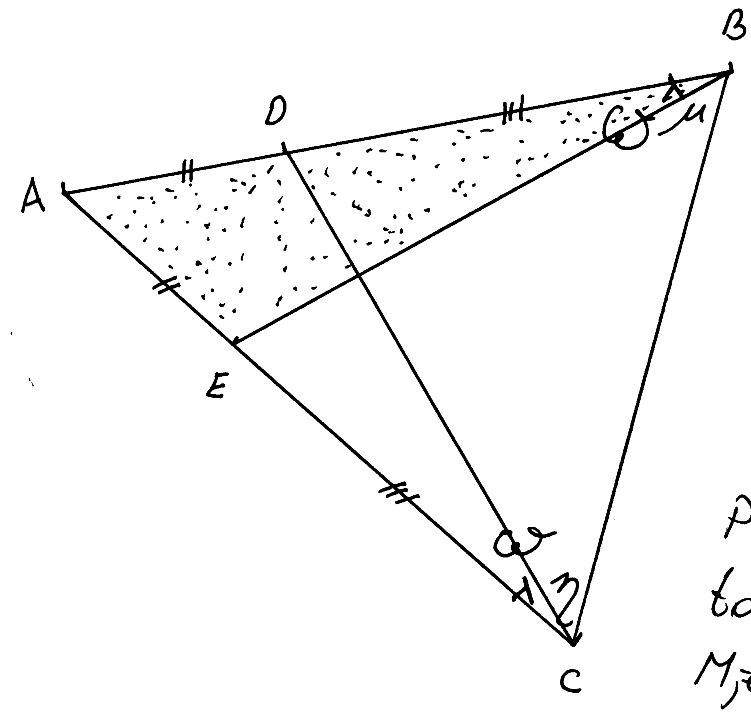
q.e.d.

# Pretpostavimo da su A, B, C, D i E tačke u neutralnoj geometriji takve da vrijedi

$$A-D-B, \quad A-E-C \quad \text{i} \quad A, B, C \text{ nisu kolinearne.}$$

Ako je $\overline{AD} \cong \overline{AE}$ i $\overline{DB} \cong \overline{EC}$ dokazati da je $\measuredangle EBC \cong \measuredangle DCB$.

Rj:




Mjere uglova $\measuredangle EBC$ i $\measuredangle DCB$ označimo redom sa $\mu$ i $\eta$.

Želimo pokazati da je

$$\mu = \eta$$

Primjetimo da kako je $\overline{AB} \cong AC$ to je $\measuredangle ACB \cong \measuredangle ABC$. Mjeru ova dva ugla označimo sa $\omega$.

Posmatrajmo $\triangle ABE$ i $\triangle ACD$.

$$\left.\begin{array}{l} \overline{AB} \cong \overline{AC} \\ \measuredangle BAE \cong \measuredangle CAD \\ \overline{AE} \cong \overline{AD} \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle ABE \cong \triangle ACD$$

$$\Downarrow$$

$$\measuredangle ABE \cong \measuredangle ACD$$

(označimo mjere ovih uglova sa $\lambda$)

Sad imamo

$$\left.\begin{array}{l} \eta = \omega - \lambda \\ \mu = \omega - \lambda \end{array}\right\} \Rightarrow \eta = \mu \Rightarrow \measuredangle EBC \cong \measuredangle DCB$$

q.e.d.

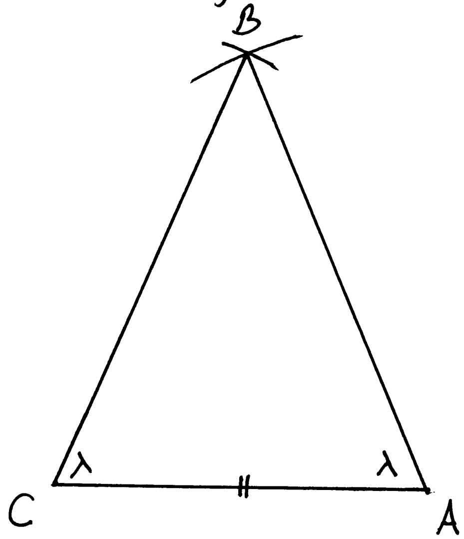
# Osnovne teoreme o podudarnosti trouglova

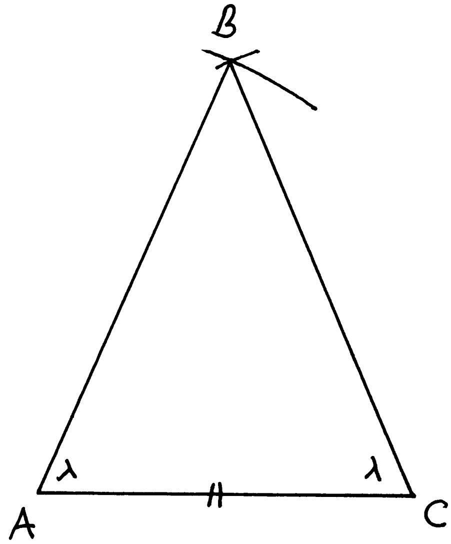
Teorem (obrat teoreme Pons Asinorum)

U neutralnoj geometriji, za dati trougao $\Delta ABC$ za koji vrijedi $\measuredangle A \cong \measuredangle C$ imamo da

$\overline{AB} \cong \overline{CB}$ i trougao je jednakokraki.

Rj: Mjere uglova $\measuredangle A$ i $\measuredangle C$ označimo sa $\lambda$. Posmatrajmo sljedeće dvije slike




Prema prethodnoj teoremi neutralna geometrija zadovoljava aksiom USU.

$$\left.\begin{array}{c} \measuredangle BCA \cong \measuredangle BAC \\ \overline{CA} \cong \overline{AC} \\ \measuredangle CAB \cong \measuredangle ACB \end{array}\right\} \overset{USU}{\Longrightarrow} \quad \Delta CAB \cong \Delta ACB$$

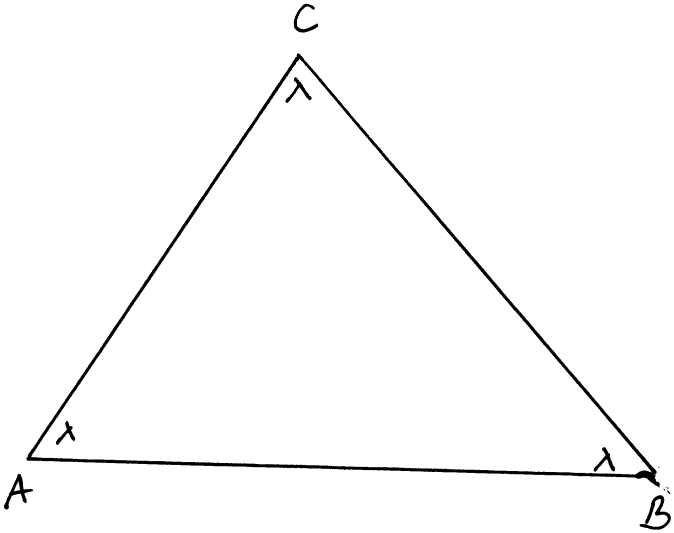
$$\Downarrow$$

$$\overline{CB} \cong \overline{AB} \Rightarrow \Delta ABC \text{ jkk}$$

q.e.d.

# Pokazati da u neutralnoj geometriji ako trougao ima sve uglove jednake tada je taj trougao jednakostranični.

Rj.




Neka je dat $\Delta ABC$.
Mjere uglova $\measuredangle A$, $\measuredangle B$ i $\measuredangle C$ označimo sa $\lambda$

$\measuredangle A \cong \measuredangle B \xRightarrow{\text{obratu teorem Pons Asinorum}} \overline{AC} \cong \overline{BC}$ ...(1)

$\measuredangle A \cong \measuredangle C \xRightarrow{\text{obrat teor. Pons Asinorum}} \overline{AB} \cong \overline{BC}$ ...(2)

(1) i (2) $\Rightarrow$ $\overline{AB} \cong \overline{AC} \cong BC$

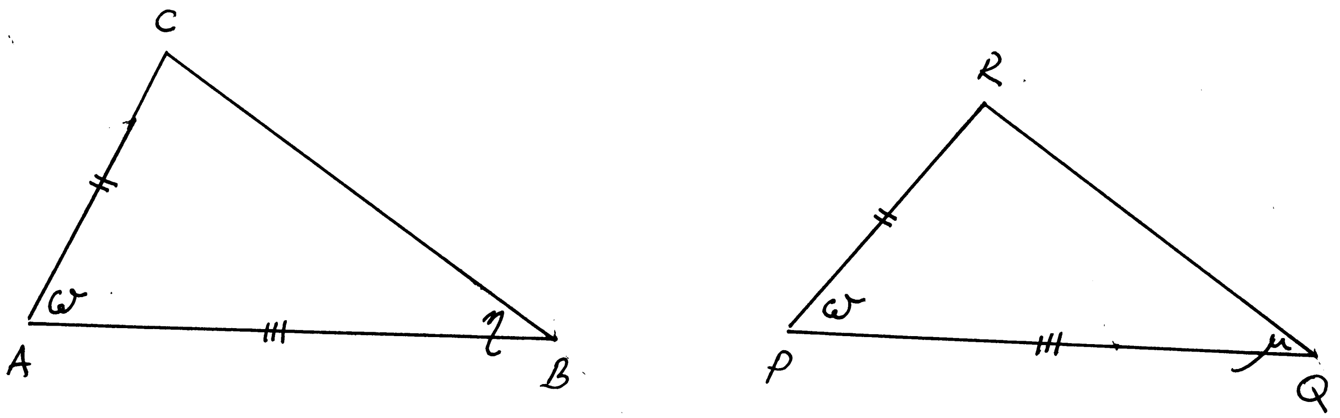
$\Downarrow$

$\Delta ABC$ jks

Teorema Ako protractor geometrija zadovoljava aksiom USU tada ona također zadovoljava aksiom SUS (pa je time neutralna geometrija).

# Dokazati teoremu iznad.

Rj. Neka su $\Delta ABC$ i $\Delta PQR$ dva trougla za koje vrijedi da je $\overline{AC} \cong \overline{PR}$, $\sphericalangle CAB \cong \sphericalangle RPQ$ i $\overline{AB} \cong PQ$. Pretpostavimo da je zadovoljena aksioma USU i pokažimo da je tada $\Delta ABC \cong \Delta PQR$ (time ćemo pokazati da je zadovoljena i aksioma SUS).




Posmatrajmo uglove $\sphericalangle ABC$ i $\sphericalangle PQR$. Za mjere ovih uglova moguć je tačno jedan od sljedeća tri slučaja

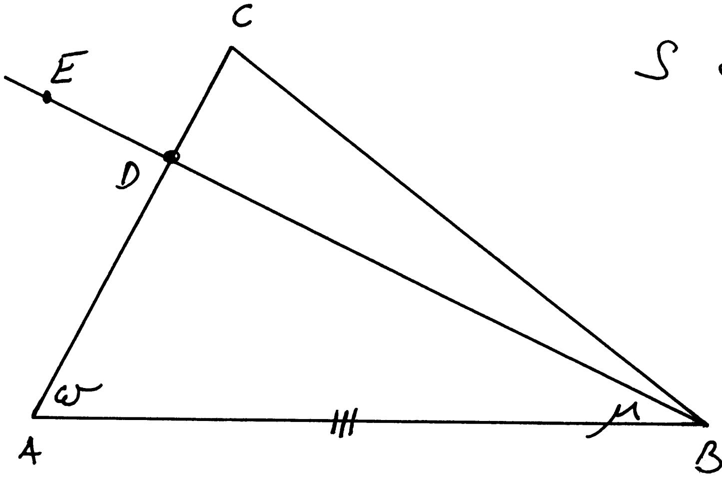
1° $m(\sphericalangle ABC) > m(\sphericalangle PQR)$

2° $m(\sphericalangle ABC) = m(\sphericalangle PQR)$

3° $m(\sphericalangle ABC) < m(\sphericalangle PQR)$

Pokažimo da slučajevi 1° i 3° nisu mogući.

Pretpostavimo da je $m(\sphericalangle ABC) > m(\sphericalangle PQR)$. Ove dvije mjere označimo sa $\eta$ i $\mu$ (1° $\eta > \mu$). Tada prema teoremi konstrukcije ugla postoji poluprava $pp[B,E) = \overrightarrow{BE}$ t.d. $m(\sphericalangle ABE) = \mu$. Neka je $\{D\} = \overrightarrow{BE} \cap \overline{AC}$. Kako je $\eta > \mu$ to je poredak $A-D-C$ (vidi sliku na sljed. strani).




$$A-D-C \Rightarrow AD < AC \quad \ldots(1)$$

S druge strane

$$\left.\begin{array}{c} \sphericalangle DAB \cong \sphericalangle RPQ \\ \overline{AB} \cong \overline{PQ} \\ \sphericalangle ABD \cong \sphericalangle PQR \end{array}\right\} \overset{USU}{\Longrightarrow} \triangle DAB \cong \triangle RPQ$$

$$\Downarrow$$

$$\overline{AD} \cong \overline{PR}$$

$$\Downarrow$$

$$AD = PR$$

$$\Downarrow \; PR = AC$$

$$AD = AC$$

#kontradikcija (sa (1))

Prema tome 1° nije moguć.
Primjetimo da bi na isti način pokazali da nije moguć 3° (umjesto $\triangle ABC$ iznad nacrtali bi trougao $\triangle PQR$).

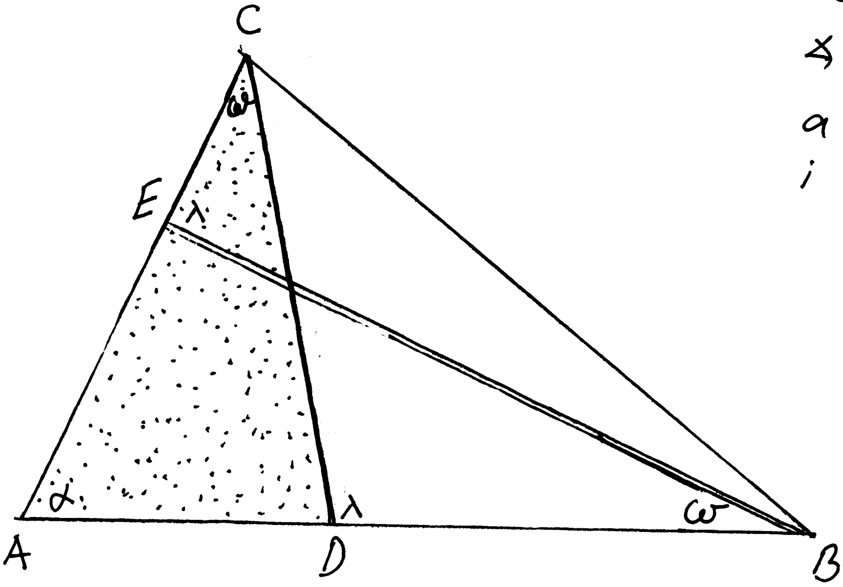
Prema tome moramo imati da je $\eta = \mu$. Sada imamo

$$\left.\begin{array}{c} \sphericalangle CAB \cong \sphericalangle RPQ \\ \overline{AB} \cong \overline{PQ} \\ \sphericalangle ABC \cong \sphericalangle PQR \end{array}\right\} \overset{USU}{\Longrightarrow} \triangle ABC \cong \triangle PQR$$

q.e.d.

(#) U neutralnoj geometriji neka je dat trougao $\triangle ABC$ za koji vrijedi $A-D-B$, $A-E-C$, $\measuredangle ABE \cong \measuredangle ACD$, $\measuredangle BDC \cong \measuredangle BEC$ i $\overline{BE} \cong \overline{CD}$. Pokazati da je $\triangle ABC$ jednakokraki (jkk).

Rj.




Označimo mjere uglova $\measuredangle ABE$ i $\measuredangle ACD$ sa $\omega$, a mjere uglova $\measuredangle BDC$ i $\measuredangle BEC$ sa $\lambda$.

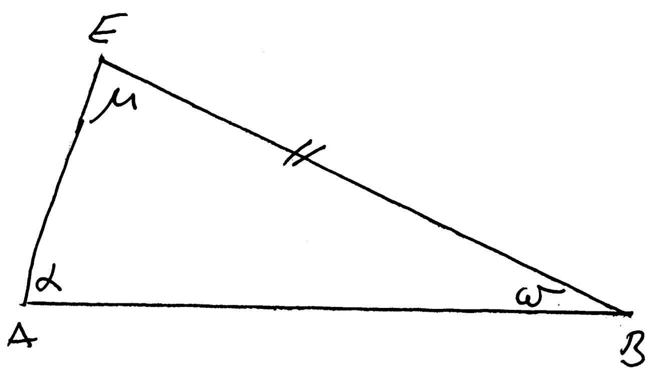
Primetimo da su $\measuredangle CEB$ i $\measuredangle BEA$ suplementarni uglovi pa je
$m(\measuredangle BEA) = 180 - \lambda$ ...(1)

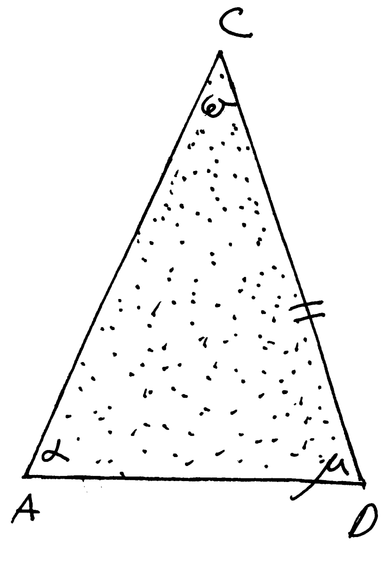
S druge strane $\measuredangle ADC$ i $\measuredangle CDB$ su suplementarni uglovi pa je
$m(\measuredangle ADC) = 180 - \lambda$ ...(2)

označimo mjere ovih uglova sa $\mu$

Na osnovu (1) i (2) $\Rightarrow$ $\measuredangle AEB \cong \measuredangle ADC$.

Posmatrajmo sada $\triangle ABE$ i $\triangle ACD$.




$$\left.\begin{array}{c} \measuredangle AEB \cong \measuredangle ADC \\ \overline{EB} \cong \overline{DC} \\ \measuredangle EBA \cong \measuredangle DCA \end{array}\right\} \overset{USU}{\Longrightarrow} \triangle ABE \cong \triangle ACD$$

$$\Downarrow$$

$$\overline{AB} \cong \overline{AC}$$

q.e.d.